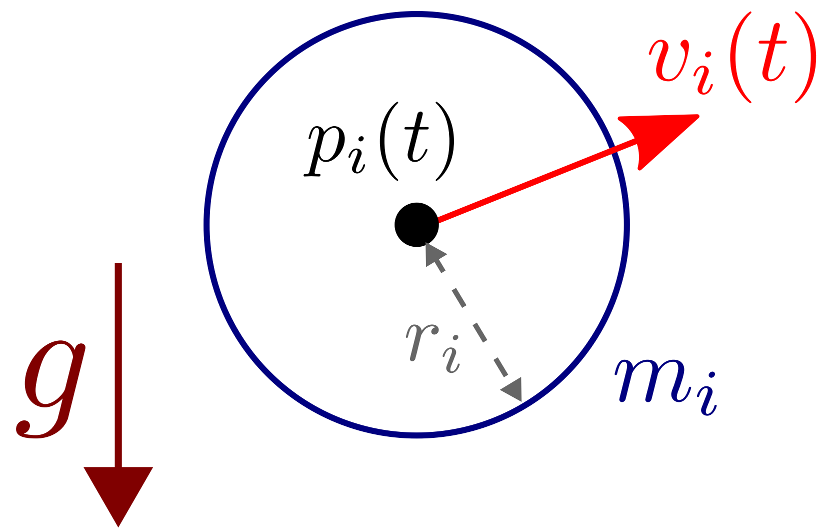
Rigid spheres

System modeling



Particles modeling the center of hard spheres.

- Spheres can collide with surrounding obstacles
- Spheres can collide with each others

- *System*: N particles with position p_i, velocity v_i, mass m_i, modeling a sphere of radius r_i.
  - Initial conditions $p_i(0) = p_i^0$, $v_i(0) = v_i^0$

- *Forces*: Single gravity forces $F_i = m_i\, g$. Collisions handled by *impulses*.

- *Temporal evolution*: Fundamental principle of dynamics $v_i(t) = p_i'(t)$, $v_i'(t) = g$

- *Numerical solution*

$$\begin{cases} v^{k+1} = v^k + h\, g \\ p^{k+1} = p^k + h\, v^{k+1} \end{cases}$$

Collision with a plane

Plane $\mathcal{P}$: parameterized using a point a and its normal n.

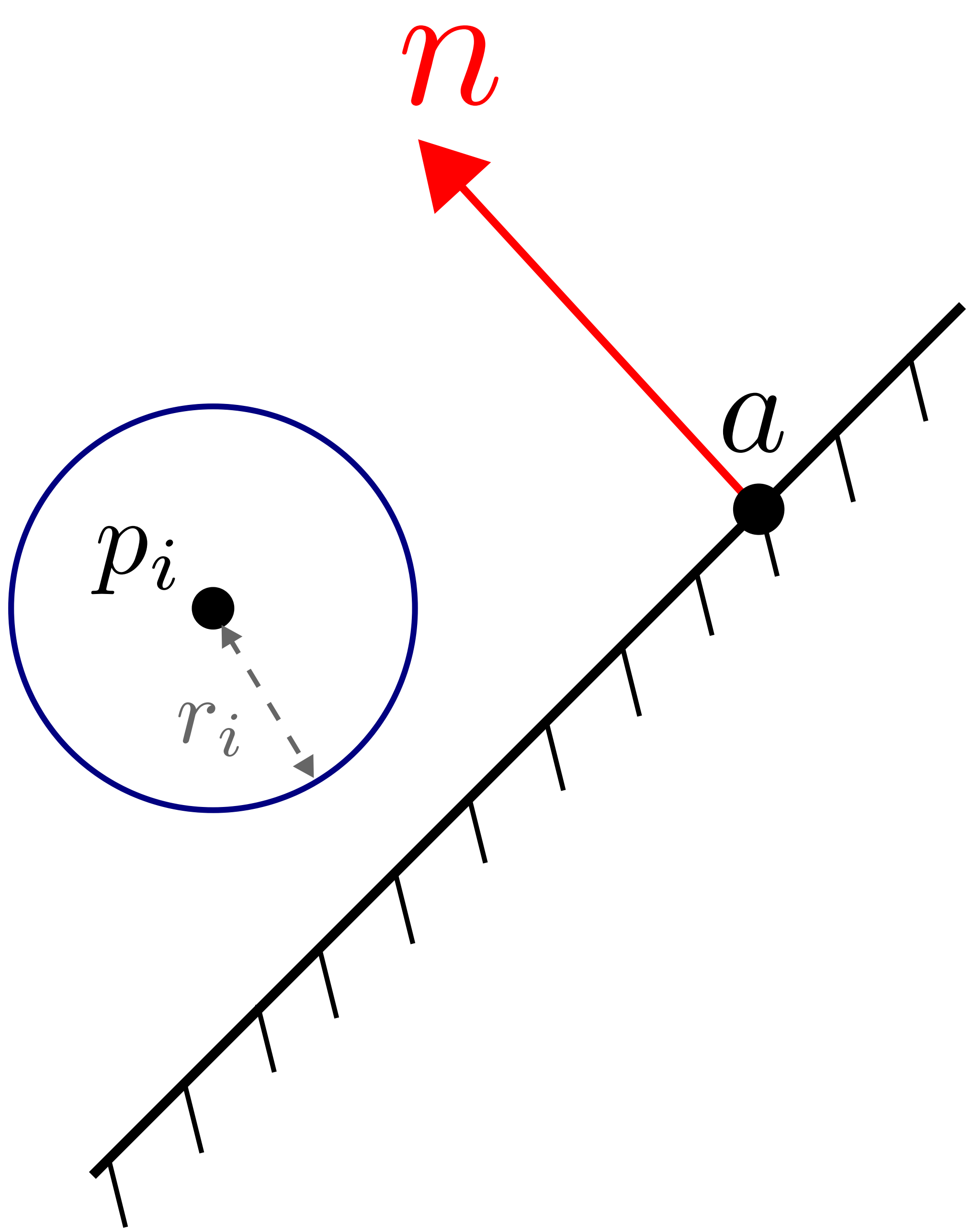
$$\{p \in \mathbb{R}^3 \in \mathcal{P} \Rightarrow (p-a)\cdot n = 0\}$$

- Sphere above plane : $(p_i - a)\cdot n > r_i$
- Sphere in collision: $(p_i - a)\cdot n \leq r_i$

- Collision detection algorithm

```
for(int i=0; i<N; ++i)
{
    float detection = dot(p[i]-a, n);
    if (detection <= r[i])
    {
        // ... collision response
    }
}
```

What should we do when a collision is detected



Collision response with plane

Suppose exact contact: $(p_i - a) \cdot n = r_i$

Collision response = **Update velocity**

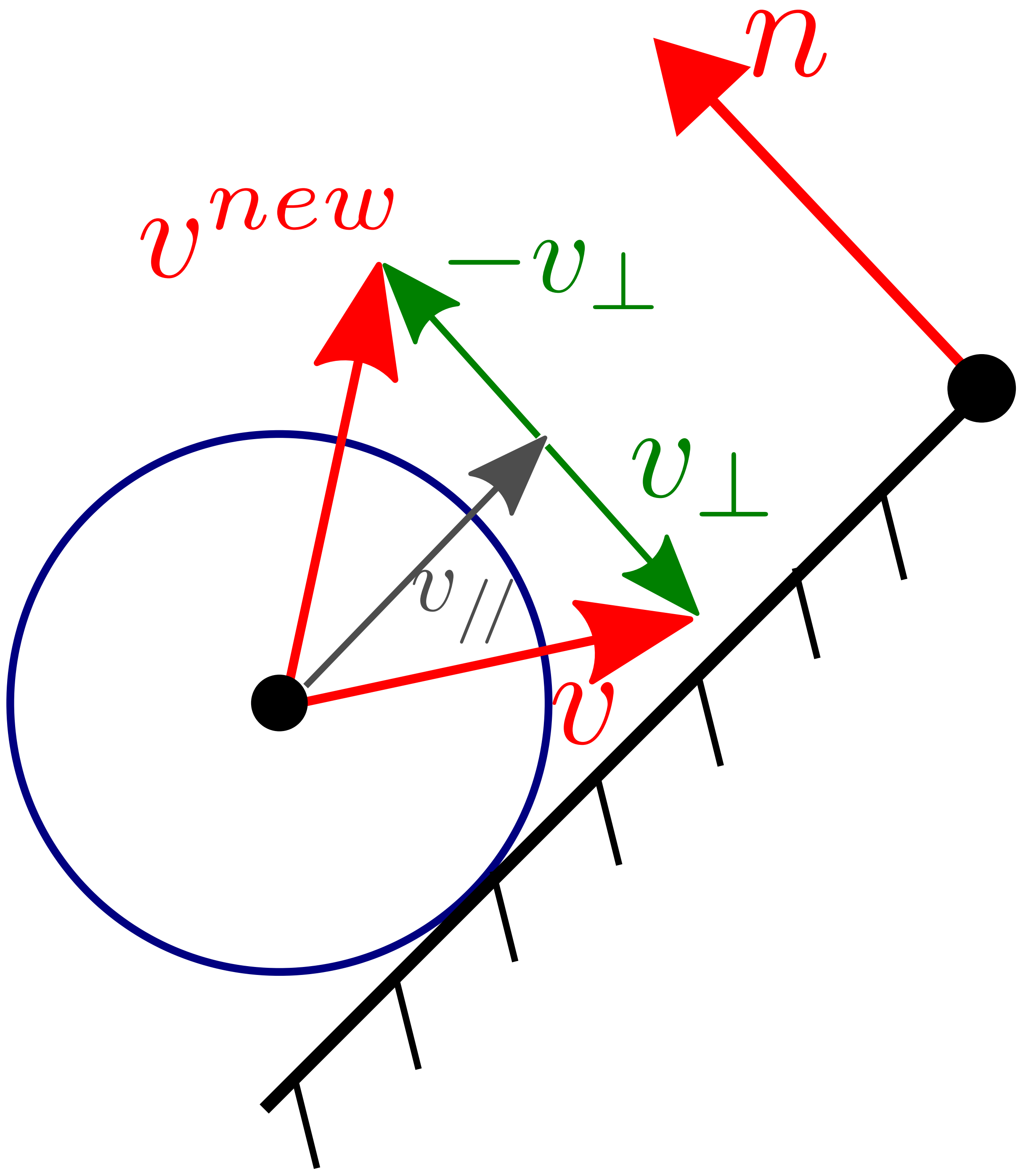
Split $v = v_{//} + v_\perp$

- $v_\perp = (v \cdot n)\, n$
- $v_{//} = v - (v \cdot n) n$

New velocity

$v^{new} = \alpha\, v_{//} - \beta\, v_\perp$

- $\alpha \in [0,1]$ Restitution coefficient in $//$ direction (friction)
- $\beta \in [0,1]$ Restitution coefficient in $\perp$ direction (impact)



Result: Collision response

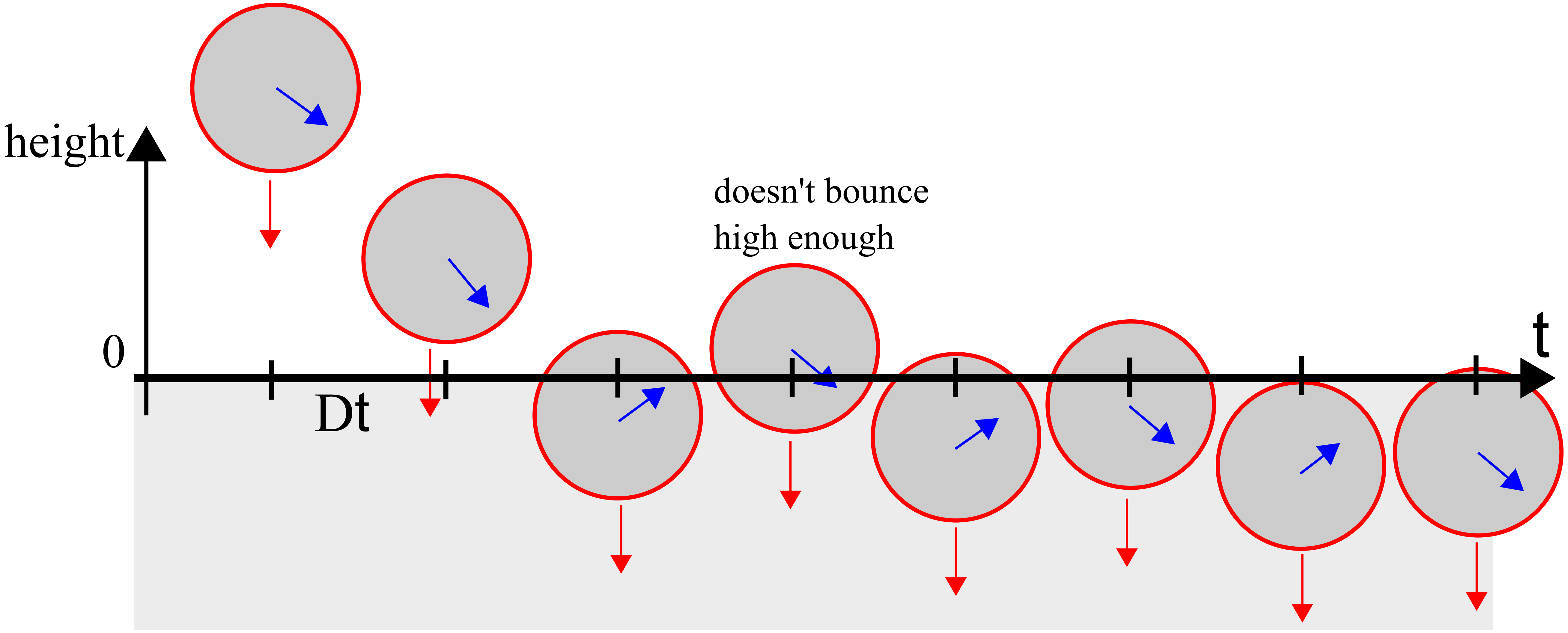
Applying collision response on speed only

Result: Collision response - issue with discrete time

We assumed contact b/w sphere and plane

But: Exact contact never happens in discrete time

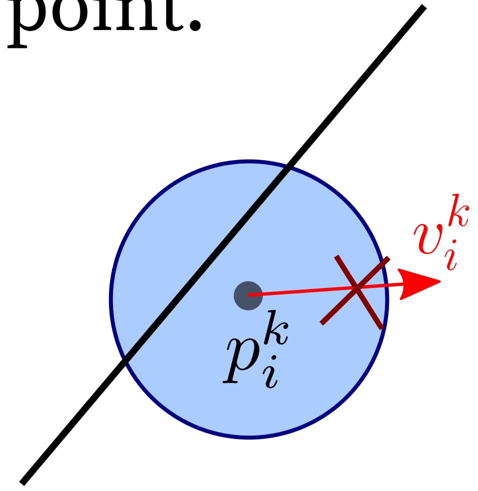
- *When collision is detected* $\rightarrow$ *already inside the wall*
- *Weight is still acting*



Collision response with plane : position

In real case (discrete time) no exact contact, but penetration $(p_i - a) \cdot n_i < r_i$

$\Rightarrow$ Need to compute collision response at contact point.

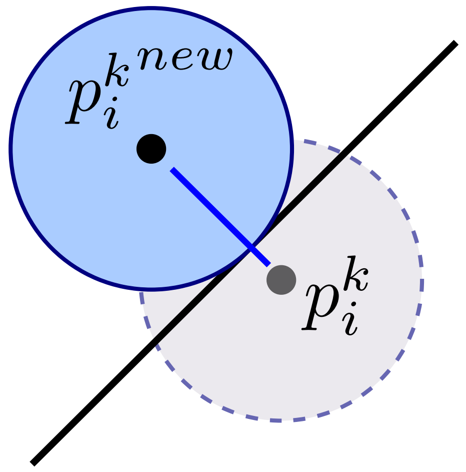


Three possibilities

(1) Update velocity to remove penetration

(+) Simple for well defined volumes

(-) Keep collision state

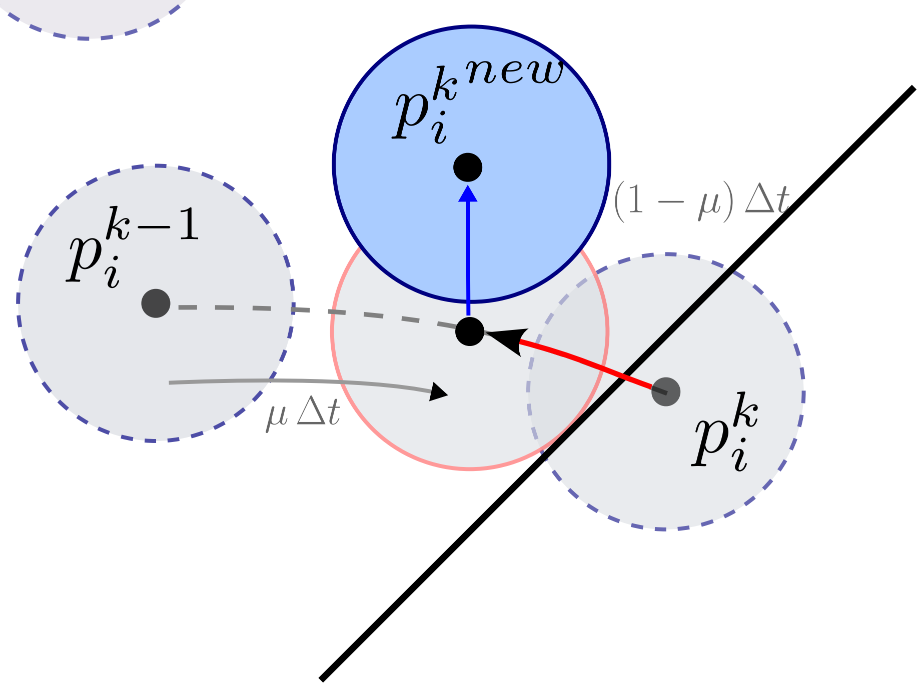


(2) Correct positions in projecting on the contact plane

Position Based Dynamics (PPD)

(+) Simple to implement

(-) Physically inaccurate



(3) Go backward in time to find exact instant of collision

Continuous Collision Detection (CCD)

(+) Physically accurate

(-) Computationally heavy (binary search, etc.)

Result: After correction

Either avoiding negative oriented velocity

Velocity bounce if $(p_i - a) \cdot n < 0$ and $v_i \cdot n < 0$

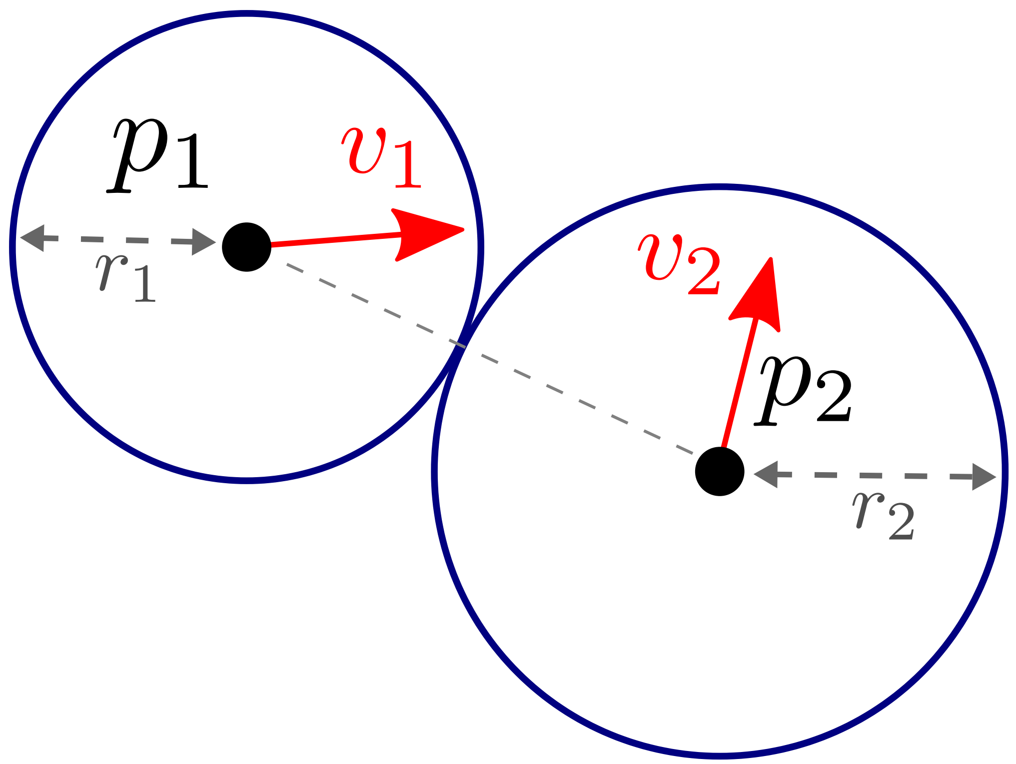
Either position projection on surface contact

$p_i^{new} = p_i + d\,n$

$d = r_i - (p_i - a) \cdot n$: distance of penetration

Collision between spheres

Given 2 spheres (p_1, v_1, r_1, m_1), (p_2, v_2, r_2, m_2).
Collision when $\|p_1 - p_2\| \leq r_1 + r_2$



What happen with their velocities ?

$v_1 \rightarrow v_1^{new}, v_2 \rightarrow v_2^{new}$

Notion of impulse

An impulse J is the integrated force over time $J = \int_{t_1}^{t_2} F(t)\,\mathrm{d}t$

$\rightarrow$ results in a sudden change of speed (/momentum) in a discrete case

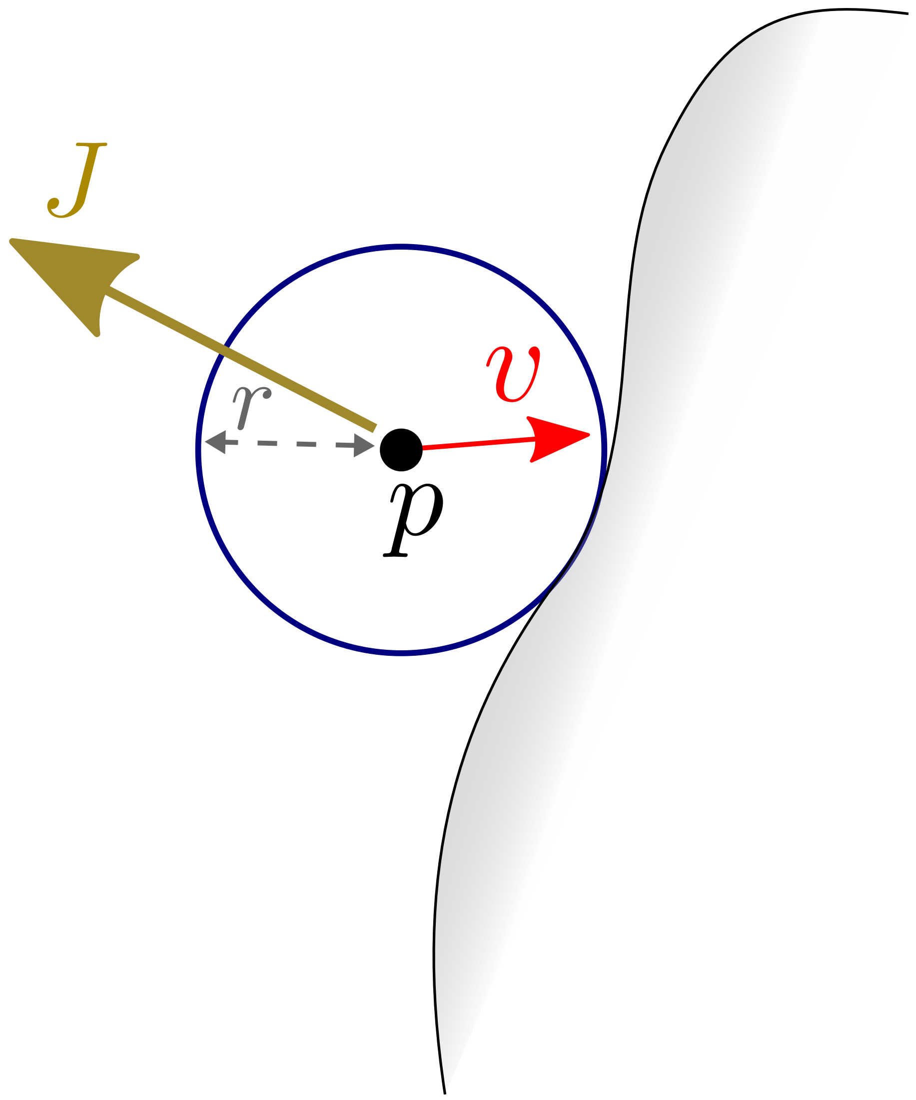
For a particle with constant mass

$$\int_{t_1}^{t_2} F(t)\,\mathrm{d}t = \int_{t_1}^{t_2} m\,a(t)\,\mathrm{d}t$$

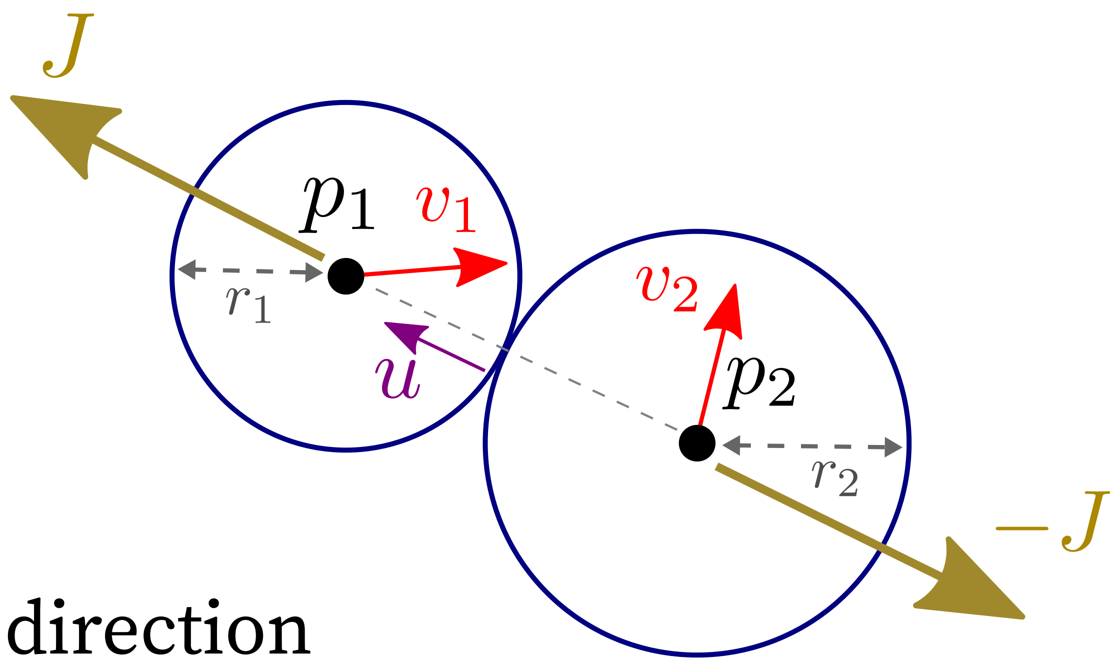
$$\Rightarrow J = m\,(v(t_2) - v(t_1))$$

For an impact $v \rightarrow v^{new}$

$$v^{new} = v + J/m$$



Two spheres in collision



Impulse orthogonal to the separating plane between the two surfaces

$$J = j\,u, \quad u = (p_1 - p_2)/\|p_1 - p_2\|$$

The system with the two spheres is preserving its linear momentum

$\Rightarrow$ Respective impulses j are equals in magnitude, and opposed in direction

$$m_1 v_1 + m_2 v_2 = m_1 v_1^{new} + m_2 v_2^{new} \Rightarrow m_1 (v_1^{new} - v_1) = -m_2 (v_2^{new} - v_2) \Rightarrow J_1 = -J_2$$

Assume collision of "hard spheres" $=$ "Elastic collision"

$=$ No loss of energy, conservation of kinetic energy of the system

$$\Rightarrow j = 2 \frac{m_1\, m_2}{m_1 + m_2} (v_2 - v_1) \cdot u$$

$$1/2\, m_1\, v_1^2 + 1/2\, m_2\, v_2^2 = 1/2\, m_1\, (v_1^{new})^2 + 1/2\, m_2\, (v_2^{new})^2$$

$$\Rightarrow m_1 v_1^2 + m_2 v_2^2 = m_1 \left(v_1 + \tfrac{j}{m_1} u\right)^2 + m_2 \left(v_2 - \tfrac{j}{m_2} u\right)^2$$

$$\Rightarrow 0 = 2\, j\; v_1 \cdot u + \tfrac{j^2}{m_1} - 2\, j\; v_2 \cdot u + \tfrac{j^2}{m_2}$$

$$\Rightarrow j = \tfrac{2}{1/m_1 + 1/m_2} (v_2 - v_1) \cdot u$$

Two spheres in collision

$$v_1^{new} = v_1 + j/m_1\, u = v_1 + 2\frac{m_2}{m_1+m_2}((v_2 - v_1)\cdot u)\, u$$
$$v_2^{new} = v_2 - j/m_2\, u = v_2 - 2\frac{m_1}{m_1+m_2}((v_2 - v_1)\cdot u)\, u$$

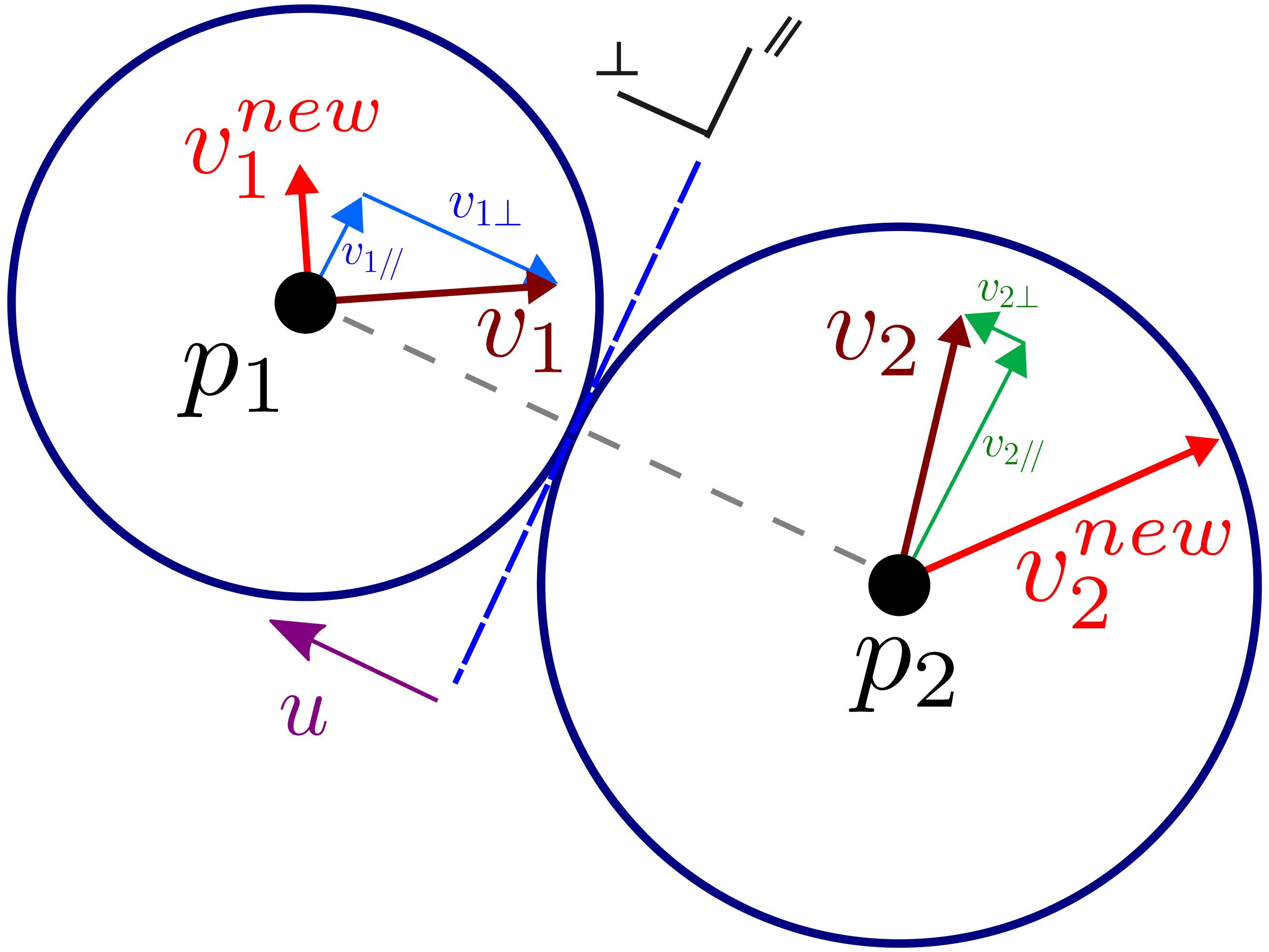
Rem. If $m_1 = m_2$: Switch their $\perp$ speeds

$$v_1^{new} = v_1 + ((v_2 - v_1)\cdot u)\, u = v_{1//} + v_{2\perp}$$
$$v_2^{new} = v_2 - ((v_2 - v_1)\cdot u)\, u = v_{2//} + v_{1\perp}$$

with $v_\perp = (v \cdot u)u$ and $v_{//} = v - (v \cdot u)u$

Can use restitution coefficient and attenuation $\alpha \in [0, 1]$

$$v_1^{new} = \alpha\,(v_{1//} + v_{2\perp})$$
$$v_2^{new} = \alpha\,(v_{2//} + v_{1\perp})$$



Handling collision between two spheres

Position Based

1. Detect collision $\|p_1 - p_2\| \leq r_1 + r_2$

If collision then:

2a. Update Velocity

Elastic collision (/bouncing)

$v_1 = \alpha\,(v_1 + j/m_1\,u)$

$v_2 = \alpha\,(v_2 - j/m_2\,u)$

2b. Correct position (project on contact surface)

$p_1 = p_1 + d/2\,u$

$p_2 = p_2 - d/2\,u$

$d = r_1 + r_2 - \|p_1 - p_2\|$: Collision depth

Velocity Based

1. Detect collision $\|p_1 - p_2\| \leq r_1 + r_2$

If collision then:

2. Update Velocity

Elastic collision (/bouncing)

If $v \cdot n < 0$

$v_1 = \alpha\,(v_1 + j/m_1\,u)$

$v_2 = \alpha\,(v_2 - j/m_2\,u)$

Summary with multiple particles

Position Based

A. Update position and velocity from field forces (gravity, friction)

B. Handle collision (velocity+position) between particles

C. Handle collision (velocity+position) with walls

(+) Good collision avoindance for the last constraints

(-) Jittering appears in stacked spheres

Velocity Based

A. Update velocity from field forces (gravity, friction)

B. Handle collision (with velocity) between particles, and walls

C. Cancel velocity component contributing to penetration

D. Update position from current velocity

(+) Smooth and stable motion

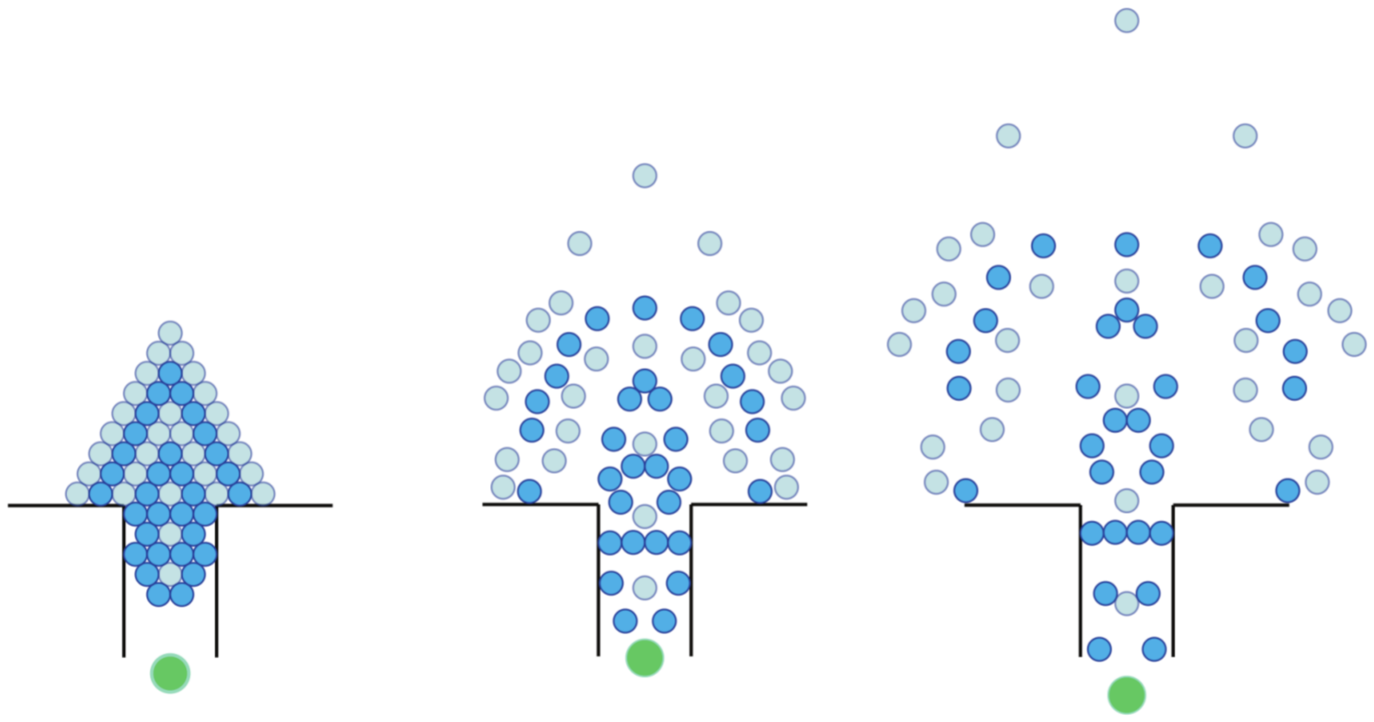
(-) Existing collision persists

Multiple collisions

Pairwise collisions $\Rightarrow$ no global collision free state

- Correcting one collision may induce new collisions.
- Order of correction does matter

Reducing time-step help, Iterating over mutliple pass help



But correct solution in all cases is complex $\longrightarrow$ global approach

- Precompute contact graph

  explicit shock propagation management

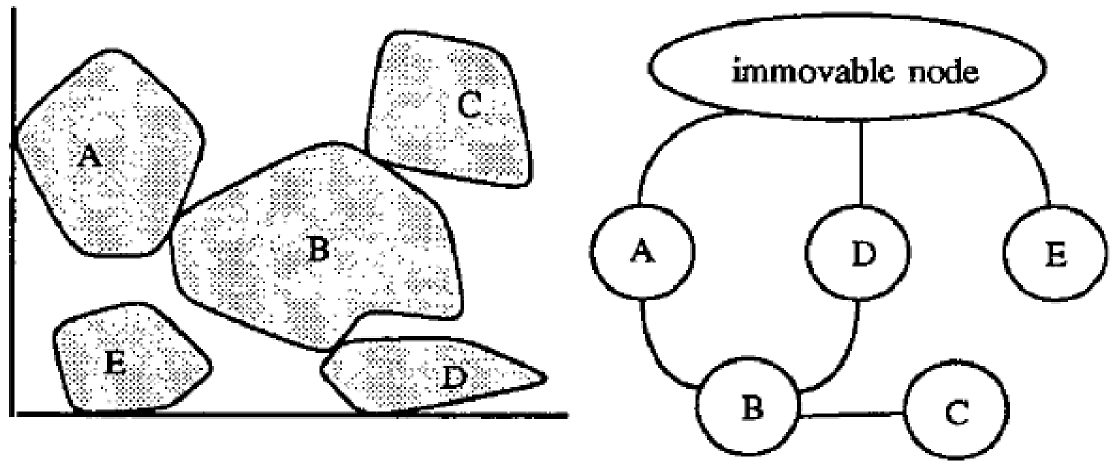
- Global constraint-based method

  Impulse: $n_i \cdot (v_i - v_j) \geq 0$

  Momentum preservation: $m_i v_i - m_j v_j = 0$

  Energy preservation/dissipation

  $\Rightarrow$ Linear Complementarity Program, Gauss Seidel, etc.



[Realistic Animation of Rigid Bodies. J. Hahn. SIGGRAPH 1988.]

[Collision Detection and Response for Computer Animation. M. Moore and J. Wilhelms. Computer Graphics 1988.]

[Reflections on Simultaneous Impact. B. Smith et al. SIGGRAPH 2012]

[Guaranteed Resolution of Simultaneous Rigid Body Impact. E. Vouga. ACM SIGGRAPH 2017]