MPRI 2.39 - Introduction to Computer Graphics
Computer Graphics main SubFields

Modeling
How to create static shapes

Animation
How to create and author time varying shapes

Rendering
How to generate 2D images from 3D data
Representing 3D shapes for Graphics Applications

**Volume representation**
- + Accurate, handle density

**Surface representation**
- + Focus on visible part
- + Fast GPU rendering, low memory footprint

=> **Computer Graphics**: Mostly focus on representing **Surfaces**
=> **Scientific visualization**: Volume data
Digital representation of surfaces
Two main representations for surfaces

Explicit representation

\[ S(u, v) = (x(u, v), y(u, v), z(u, v)) \]

Parametric map

Implicit representation

\[ S = \{ (x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0 \} \]

Isosurface of scalar field

+ Neighborhood information

+ Topological modification
Two main representations for surfaces

Example for a sphere

Explicit representation

\[ S(u, v) = (x(u, v), y(u, v), z(u, v)) \]

Parametric map

\[ S(u, v) = \begin{cases} 
  x(u, v) = R \sin(u) \cos(v) \\
  y(u, v) = R \sin(u) \sin(v) \\
  z(u, v) = R \cos(u) 
\end{cases} \]

Implicit representation

\[ S = \{ (x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0 \} \]

Isosurface of scalar field

\[ F(x, y, z) = x^2 + y^2 + z^2 - R^2 \]
Difficulty of surface representation using function

Which function can represent this shape?

\[ S(u,v) = ? \]

\[ F(x,y,z) = ? \]
Objective of surface representation

Main Idea => Use of **piecewise approximation**

Ideal surface representation
- **Approximate** well any surface
- Require **few samples**
- Can be **rendered** efficiently (GPU)
- Can be manipulated for **modeling**

Example of models:
- **Mesh-based**: Triangular meshes, Polygonal meshes, Subdivision surfaces
- **Polynomial**: Bezier, Spline, NURBS
- **Implicit**: Grid, Skeleton based, RBF, MLS
- **Point sets**

=> For projective/rasterization render pipeline: always render **triangular meshes** at the end

  + Simplest representation
  + Fit to GPU Graphics render pipeline
  - Requires large number of samples: complex modeling
  - Tangential discontinuities at edges
Meshes

Simplest possible representation of 3D surfaces: set of triangles
Described as a triplet: (Vertices, Edges, Faces)
\[ S = (\mathcal{V}, \mathcal{E}, \mathcal{F}) \]

- **Vertex**
  \[ \mathcal{V} = (v_1, \ldots, v_N) \]

- **Edge**
  \[ \mathcal{E} = (v_1, \ldots, v_{N_e}) \subseteq (\mathcal{V}^2)^{N_e} \]

- **Face**
  \[ \mathcal{F} = (f_1, \ldots, f_{N_f}) \subseteq (\mathcal{V}^3)^{N_f} \]
Mesh encoding

Exemple for a tetrahedron

- 1st Solution: *Soup of polygons*

  \[\text{triangles} = [(0.0, 0.0, 0.0), (1.0, 0.0, 0.0), (0.0, 0.0, 1.0),
  (0.0, 0.0, 0.0), (0.0, 0.0, 1.0), (0.0, 1.0, 0.0),
  (0.0, 0.0, 0.0), (0.0, 1.0, 0.0), (1.0, 0.0, 0.0),
  (1.0, 0.0, 0.0), (0.0, 1.0, 0.0), (0.0, 0.0, 1.0)]\]

- 2nd solution: *Geometry, Connectivity*

  \[\text{geometry} = [(0.0, 0.0, 0.0), (1.0, 0.0, 0.0), (0.0, 1.0, 0.0), (0.0, 0.0, 1.0)]\]
  \[\text{connectivity} = [(0,1,3), (0,3,2), (0,2,1), (1,2,3)]\]

=> Prefered solution
  + more space efficient
  + modifying 1 vertex = 1 operation
Mesh buffer encoding in C++

```cpp
#include <vector>
#include <array>

struct vec3 { float x, y, z; }

using index3 = std::array<unsigned int, 3>;

int main()
{
    std::vector<vec3> geometry = {
        {0.0f, 0.0f, 0.0f}, {1.0f, 0.0f, 0.0f},
        {0.0f, 1.0f, 0.0f}, {0.0f, 0.0f, 1.0f}
    };
    std::vector<index3> connectivity = {
        {0, 1, 3}, {0, 3, 2}, {0, 2, 1}, {1, 2, 3}
    };

    return 0;
}
```
Example of 3D Mesh file

<table>
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<tr>
<th>v</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
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</tbody>
</table>

Open question: How to display it efficiently on screen?
How to render surface on a screen
How to render surfaces

Ray tracing

- Throw rays from light-sources/camera
- Intersect rays with 3D shapes
- Pixel-wise computation

+ Photo-realistic rendering
  (Soft shadows, reflection, caustics)
+ Handle general surfaces
- High computational cost

=> Restricted to offline rendering (but developing more and more)

Projection/Rasterization

- Assume shapes made of triangles
  1. Project each triangle onto camera screen space
  2. Rasterize projected triangle into pixels
- Triangle-wise computation

+ Efficiently implemented on GPU
- Limited to triangles
- No native effects (shadows, transparency, etc)

=> The standard real time rendering with GPU
Projection/Rasterization

Object made of **triangles only**

1. **Project vertices of triangles**
   - Projection computed as matrix operation (projective space $p' = Mp$)

2. **Rasterization**
   - Discrete geometry
   - Interpolate attributes (colors, etc) on each pixel
   
   => At interactive frame rate ($\geq 25$ fps)
   - Project all triangles of shapes
   - Fill all pixels of each projected triangle
Quick fundamental notions for practical 3D programming

- Affine transform as 4D matrices
- Perspective and projective space
- Illumination and normals
Affine transforms and 4D vectors/matrices

Preliminary note
- We use a lot affine transformations to place shapes in 3D space
  Translation, Rotation, Scaling
- In CG vectors are often expressed in 4D, and matrices are $4 \times 4$.

=> Reason: Affine transforms can be expressed linearly (with matrices) in 4D

\[
p = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \quad M = \begin{pmatrix} m_{00} & m_{01} & m_{02} & t_x \\ m_{10} & m_{11} & m_{12} & t_y \\ m_{20} & m_{21} & m_{22} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]
Affine transform in 2D

General principle in the 2D case

Example for a point \( p = (x, y) \)

Rotation \( R = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \), Scaling \( S = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \), Translation \((x + t_x, y + t_y)\) (not linear)

Cannot express conveniently composition b/w several rotation, scaling, translation.

Trick - Add an extra coordinates to points \( p = (x, y, 1) \) (homogeneous coordinates).

Then translation can be expressed linearly \( p' = T \cdot p \), with \( p' = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix} \)

Similarly with rotation \( R = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \), and scaling \( S = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \).
Affine transform matrix

With the extra dimension (in 2D):
Translation $T$, rotation $R$, scaling $S$ can be composed as matrix products representation

\[
M = T_0 R_0 S_0 T_1 R_1 S_1 \ldots \quad M = \begin{pmatrix}
m_{00} & m_{01} & t_x \\
m_{10} & m_{11} & t_y \\
0 & 0 & 1
\end{pmatrix}.
\]

$m_{ij}$ : linear part (rotation and scaling); $t_{x/y}$ : translation part

Similar in 3D but with 4-components vectors, and 4 × 4 matrices.

\[
p = (x, y, z, 1) - \text{represents 3D position}
\]
\[
M = \begin{pmatrix}
m_{00} & m_{01} & m_{02} & t_x \\
m_{10} & m_{11} & m_{12} & t_y \\
m_{20} & m_{21} & m_{22} & t_z \\
0 & 0 & 0 & 1
\end{pmatrix} - \text{represents 3D affine transformation (rotation, scaling, translation)}
\]

Note: vectors and points can be expressed
- 3D point $(x, y, z, 1)$ - translation applies.
- 3D vector $(x, y, z, 0)$ - translation doesn't apply.
Perspective and projective space

Modeling perspective projection requires division.

ex. in 2D (1D projection)

\[ y' = x' \frac{y}{x} = f \frac{y}{x} \quad (f: \text{focal}) \]

Linear model using 3D vectors in projective space.

\[
\begin{pmatrix}
\begin{bmatrix}
f \\
fx \\
x
\end{bmatrix}
\end{pmatrix}
\begin{pmatrix}
f \\
fy \\
x
\end{pmatrix}
= \begin{pmatrix}
f & 0 & 0 \\
0 & f & 0 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
1
\end{pmatrix}
\]

considering that the last coordinate must always be normalized to 1 (for points).

**Projective space**

- *Real* points lie on \( z = 1 \)
- Vectors lie on \( z = 0 \)

Real coordinates of points are obtained after normalization (division by \( z \)).
Perspective matrix

Perspective space: Allows perspective projection expressed as matrix.

Common constraints (in OpenGL)
- Wrap the viewing volume (truncated cone with rectangular basis called frustum) \((z_{near}, z_{far}, \theta)\) to a cube.
- \(\theta\): view angle
- \(p = (x, y, z, 1) \in \text{frustum} \Rightarrow p' = (x', y', z', 1) \in [-1, 1]^3\).

In practice

=> You must define \(z_{near}, z_{far}\)

=> \(z_{far} - z_{near}\) should be as small as possible for maximum depth precision.

To which view space coordinates are mapped 3D world space points at \(z_{near}, z_{far}\)?
Per-vertex normal and illumination

For smooth looking meshes, we define a normal per-vertex.

- Vertices are seen as samples on a smooth underlying surface

- Normals are used for illumination  
  *ambient, diffuse, specular components*

- Phong shading interpolates normals on each  
  *fragments* of triangles, and compute illumination.

Possible automatic computation of normals: averaged normals of surrounding triangle.

\[ n_k = \frac{\sum_{j \in V_k} \hat{n}_j}{\| \sum_{j \in V_k} \hat{n}_j \|}, \quad V_k: \text{neighboring triangles.} \]