MPRI 2.39 - Surface representation
Representing surfaces

- **Meshes** *(saw in previous class)*
- Parametric Spline
- Subdivision
- Point set
Mesh: pro cons

**Pros**

The simplest representation
The most general (can approximate any sampled surface)
The only natively handled by GPU (fast rendering)

**Cons**

$C^0$ only: The worst approximation for smooth surface.
⇒ High number of samples; complex modeling
Representing surfaces

- Meshes
- **Parametric Spline**
- Subdivision
- Point set
Objective of Parametric Spline

Triangle: simplest linear element

Parametric spline objectives
- Allow higher order element (less samples)
- Fully defined by positions (modeling purpose)
- $C^2$ smoothness (tangential + curvature continuity)
  Allows smooth reflexion
**Parametric Spline for 1D curves**

General form: $c(s) = \sum_{k} b_k(s) \ p_k$

$b_k$: basis functions (smooth polynomial, minimal support)
$P_k$: control points (user defined)

**Straight segment**

$c(s) = (1 - s) \ p_0 + s \ p_1$

**Cubic uniform B-spline**

$c(s) = \sum_{k=0}^{3} b_k(s) \ p_k$

\[
b_{i,k+1}(s) = \frac{s-t_i}{t_{i+k+1}-t_i} b_{i,k}(s) + \frac{t_{i+k+1}-s}{t_{i+k+1}-t_i} b_{i+1,k}(s)
\]

$b_{i,1}(s) = 1 \ t_i \leq s \leq t_{i+1} \ , \ 0 \ otherwise$

\[
\begin{align*}
  b_0(s) &= (1 - s)^3 / 6 \\
  b_1(s) &= (3s^3 - 6s^2 + 4) / 6 \\
  b_2(s) &= (-3s^3 + 3s^2 + 3s + 1) / 6 \\
  b_3(s) &= s^3 / 6 
\end{align*}
\]
Parametric Spline for Surfaces

*Standard approach:* Tensor product of polynomials

**Curve**

\[ c(s) = \sum_{k} b_k(s) \quad p_k \]

**Surface**

\[ S(u, v) = \sum_{i} \sum_{j} b_i(u) b_k(v) \quad p_{ij} \]
Parametric Spline for Surfaces, matrix formulation

Curve
\[ c(s) = \sum_k b_k(s) p_k \]
\[ c(s) = \begin{pmatrix} s^3 & s^2 & s & 1 \end{pmatrix} M \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \]

Surface
\[ S(u, v) = \sum_i \sum_j b_i(u) b_k(v) p_{ij} \]
\[ S(u, v) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} M \begin{pmatrix} p_{00} & p_{01} & p_{02} & p_{03} \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ p_{30} & p_{31} & p_{32} & p_{33} \end{pmatrix} M^t \begin{pmatrix} v^3 \\ v^2 \\ v \\ 1 \end{pmatrix} \]

\[
M = \frac{1}{6} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}
\]

fully defined by the basis functions.
Parametric Spline for 1D curves

**Pro**
- $C^2$ smoothness
- Derivative/curvature computation

**Cons**
- Limited to $4 \times 4$ grid based patch topology

*How to merge surfaces, add a branch, etc*
Representing surfaces

- Meshes
- Parametric Spline
- **Subdivision**
- Point set
Objective of Subdivision surface

- Keep advantages from **general topology** of meshes
- Smoothness of parametric splines

Example on curves

For **surfaces**: 2D subdivision mask

\[
\begin{pmatrix}
p_{2k}^{n+1} \\
p_{2k+1}^{n+1}
\end{pmatrix} =
\begin{pmatrix}
1/3 & 2/3 \\
2/3 & 1/3
\end{pmatrix}
\begin{pmatrix}
p_{k}^{n+1} \\
p_{k+1}^{n+1}
\end{pmatrix}
\]
Exemple of Subdivision Surface: Doo-Sabin

Given a face \((p_i)_i = [0, N - 1]\)

- Compute middle vertex \(m_i = (p_i + p_{i+1})/2\)
- Compute barycenter of the face \(b = 1/N \sum_{i=0}^{N-1} p_i\)
- New vertices are \(n_i = (p_i + m_i + m_{i-1} + b)/4\)
Subdivision surfaces

Pros
- Arbitrary input mesh topology
- Guaranteed convergence to $C^1$ or $C^2$ smooth surface.

Cons
- Final shape is hard to predict (depends on scheme, connectivity)
- No direct access to differential values
Representing surfaces

- Meshes
- Parametric Spline
- Subdivision
- Point set
Point set

Raw 3D point sets are widely available from 3D scanners
- No explicit connectivity
- Huge amount of points \((10^7-10^{10})\)

Idea: Represent entire surfaces from point sets only
Point set rendering

Surface looking from point sets only requires specific rendering

$\Rightarrow$ **Surface splatting**
- Cover surface by *surfels* - *oriented disks tangent to surface*
- Render them in image space as ellipsoids

In 3D space

Render in image space
MPRI 2.39 - Smooth Parametric Surfaces and Differential Geometry
Parametric smoothness

Let \( S \) be mapping of a surface \( S : \begin{cases} D \subset \mathbb{R}^2 & \to \mathbb{R}^3 \\ (u, v) & \mapsto S(u, v) \end{cases} \)

- \( S \) is \( C^1 \) if \( S_u \) and \( S_v \) are defined and continuous
- \( S \) is \( C^2 \) if \( S_{uu}, S_{uv}, S_{vv} \) and \( S_{uv} \) are defined and continuous

\[
S_u = \frac{\partial S}{\partial u}, \quad S_{uv} = \frac{\partial^2 S}{\partial u \partial v}
\]
Geometric smoothness

Let $\Gamma$ be the image of the mapping $S$.

- $\Gamma$ is $G^1$ it has a tangent plane in every position
- $\Gamma$ is $G^2$ it has curvature in every position

Note
- $S \subset C^k \neq \Gamma \subset G^k$
- $G^2$ is required for smooth reflexion within the surface.
Parametric vs Geometric smoothness example

Are these functions $C^1, C^2, G^1, G^2$?

\[ f(t) = \begin{cases} (t,t) & , \ t \in [-1,0] \\ (t/2,t/2) & , \ t \in [0,2] \end{cases} \]

\[ h(t) : \begin{cases} x(t) = R(t - \sin(t)) \\ y(t) = R(1 - \cos(t)) \end{cases} \]
Reminder about parametric curve

Curve $S(t)$, $S : \begin{cases} \mathbb{R} \rightarrow \mathbb{R}^3 \\ t \rightarrow (S_x(t), S_y(t), S_z(t)) \end{cases}$

- Tangent vector at position $S(t)$: $T(t) = S'(t)/\|S'(t)\|$

- Length of the curve $L = \int_t \|S'(t)\| \, dt$
Curvature of a curve

**Geometric interpretation**
- Radius of curvature $r$ : radius of the osculating circle.
- Curvature $\kappa = 1/r$.

**Definition** Consider the curvilinear parameterization $\mu$ such that $\|S'(\mu)\| = 1$.
- $|\kappa(\mu)| = \left\| \frac{\partial T}{\partial \mu}(\mu) \right\|$: rate of change of the tangent vector
- $\kappa$ is considered positive/negative with respect to the direction of the normal vector.

**Computational definition**
$$|\kappa(t)| = \frac{\|S'(t) \times S''(t)\|}{\|S'(t)\|^3}$$
Curvature of a surface: intuition

Keenan Crane, Digital Geometry Processing with Discrete Exterior Calculus. ACM SIGGRAPH 2013
Tangent Plane and Normal to a Surface

- Parametric Surface $S(u, v) = (S_x(u, v), S_y(u, v), S_z(u, v))$ with image $\Gamma$.

- Tangent space of $S$ at $p = S(u, v)$
  
  $$T_p(S) = \{ S_u(u, v) h_u + S_v(u, v) h_v \mid (h_u, h_v) \in \mathbb{R}^2 \}$$

- Unit normal at parameter $(u, v)$: $n(u, v) = (S_u \times S_v)/\|S_u \times S_v\| \in \mathbb{S}^2$

- The Gauss map $N$
  
  $$N(p) = \begin{cases} 
  \Gamma & \rightarrow \mathbb{S}^2 \\
  p = S(u, v) & \rightarrow N(p) = n(u, v) 
  \end{cases}$$
Integral properties

- Area $A$ of $\Gamma$: $A = \iint_{(u,v) \in \mathcal{D}} \| S_u \times S_v \| \, du \, dv$

- Volume $V$ defined by $\Gamma$: $\forall e \in S^2$, $V = \iiint_{(u,v) \in \mathcal{D}} < S(u,v), e > < n(u,v), e > \, du \, dv$

- Definition of the Volume $V = \iiint_{\Omega} d\Omega$
- Green-Ostrogradsky / Divergence theorem $\iiint_{\Omega} \text{div}(f) \, d\Omega = \int_S < f, n > \, dS$
- We set $f(p) = < p, e > e$ for any $e \in S^2$
  \( (\text{div}(f) = \|e\|^2 = 1) \)

$\Rightarrow V = \iiint_{\Omega} 1 \, d\Omega = \int_S < S(p), e > e, n > dS = \int_S < S(p), e > < n, e > dS$
First fundamental form

- 2D curve $C \subset D$
  \[ C(t) = (C_x(t), C_y(t)) \]

- Length of $C$ $L = \int_t (C''^T(t) C'(t))^{1/2} \, dt$

- 3D curve $C_s = S \circ C$
  \[ C_s(t) = (S_x(C_x(t), C_y(t)), S_y(C_x(t), C_y(t)), S_z(C_x(t), C_y(t))) \]

- Length of 3D curve $L_s$
  \[ L_s = \int_t ((S \circ C)'^T(t) (S \circ C)'(t))^{1/2} \, dt \]
  \[ L_s = \int_t (C''^T(t) I_S(t) C'(t))^{1/2} \, dt \]
  $I_S(t)$: first fundamental form
First fundamental form derivation

\[- L_s = \int_t ( (S \circ C)'^T (t) (S \circ C)' (t) )^{1/2} \, dt \]

\[- (S \circ C)' = C'_x (S, u \circ C) + C'_y (S, v \circ C) \]

\[- (S \circ C)' = (S, u \circ S, v ) \begin{pmatrix} C'_x \\ C'_y \end{pmatrix} = \partial S^T \, C' \]

\[- (S \circ C)'^T (S \circ C)' = (C''^T \partial S) (\partial S^T \, C') = C''^T \, I_s \, C'' \]

\[I_s = \partial S \, \partial S^T \]
First fundamental form properties

$I_S$: first fundamental form / metric tensor

$$I_S = \begin{pmatrix} S^2_{,u} & \langle S_{,u}, S_{,v} \rangle \\ \langle S_{,u}, S_{,v} \rangle & S^2_{,v} \end{pmatrix}$$

$\sqrt{\det(I_S)}$: Local change of area

Area of $\Gamma = \iint_{(u,v) \in \mathcal{D}} \sqrt{\det(I_S)} \, du \, dv$
Second fundamental form

Consider $S$ to be locally defined in the local tangent plane.

The Taylor expansion of $S$ along the normal direction, at the second order is

$$z(u, v) = \frac{1}{2} \langle S_{uu}, n \rangle \, du^2 + \frac{1}{2} \langle S_{vv}, n \rangle \, dv^2 + \langle S_{uv}, n \rangle \, du \, dv$$

We call the second fundamental form the associated bilinear form

$$\Pi_S = \begin{pmatrix} \langle S_{uu}, n \rangle & \langle S_{uv}, n \rangle \\ \langle S_{uv}, n \rangle & \langle S_{vv}, n \rangle \end{pmatrix}$$
Weingarten map and Curvature

- $W_S = \Pi_s I_s^{-1}$ is the Weingarten matrix (or shape operator - differential of the Gauss map).

- $W_S$ is diagonalizable and has real eigenvalues. $W_S = V^T \Lambda V$

$\Lambda = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$, $(\kappa_1, \kappa_2)$: two principal curvatures

Eigenvectors are the principal directions of curvature (in the local tangent plane).
Local surfaces types

A surface can, locally, be

- Planar: $\kappa_1 = \kappa_2 = 0$

- Cylindrical: $\kappa_1 \kappa_2 = 0, \kappa_1 \neq 0$ or $\kappa_2 \neq 0$

- Elliptic: $\kappa_1 \kappa_2 > 0$

- Hyperbolic: $\kappa_1 \kappa_2 < 0$

Keenan Crane, Digital Geometry
Processing with Discrete Exterior Calculus. ACM SIGGRAPH 2013
Mean and Gauss curvatures

- Gauss curvature: \( K_S = \kappa_1 \kappa_2 = \det(W_S) = \frac{\det(\Pi_S)}{\det(I_S)} \)

- Mean curvature: \( H_S = \frac{1}{2} (\kappa_1 + \kappa_2) = \frac{1}{2} \tr(W_S) \)

- \( H_S = 0 \iff S \) is a minimal surface
- \( K_S = 0 \iff S \) is a developable surface